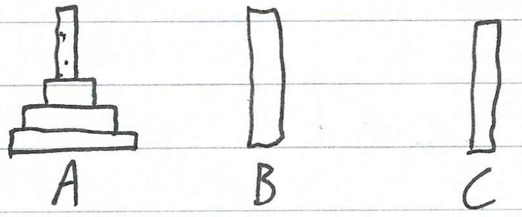
Problem Sheet 3

Q1 Hanoi



Move from A to C.

$n = 1 \Rightarrow 1$ $A \to C$
$n = 2 \Rightarrow 3$ $A \to B$, $A \to C$, $B \to C$
$n = 3 \Rightarrow 7$ $A \to C$, $A \to B$, $C \to B$, $A \to C$, $B \to A$, $B \to C$, $A \to C$

Rules:
- Smallest to C, next to B, C to B, A to C, B to C, B to A, C to A gets $n-2$ from pile into correct location.

Copied in — don't understand!
Need to establish a cost function.

Model tower as n high. Cost:

$n-1$	from	A	to	B	has cost	C_{n-1}
1		A		C		1
$n-1$		B		C		C_{n-1}

Moving 1 disk takes 1 move.

$C_1 = 1$

$C_n = 2C_{n-1} + 1$

$C_n = X_n - c$

$X_n - c = 2X_{n-1} - 2c + 1$

Cancel a c

$X_n = 2X_{n-1} - c + 1$

Set $c = 1$ to cancel out c

Use ansatz $X_n = A\lambda^w$

$A\lambda^n = 2A\lambda^{n-1}$

Cancel As

$\lambda^n = 2\lambda^{n-1}$

Set $\lambda = 2$ (as $X_n = A2^w$)

$C_n = A2^w - 1$

$C_1 = A2 - 1 = 1$

$2A = 2$

$A = 1$

$C_n = 2^n - 1$

Need to revisit as don't understand how this works.

Q2 Recursion Relations

Classification –

Linear – linear function of preceding terms

Non-linear – scaling of previous terms.

Temporal Dynamics –

Stability $|f'(x^{stat})|$

<1 stable
$=1$ marginal
>1 unstable
$=0$ superstable

→ $X_n = X_{n-1} + X_{n-2} + X_{n-4}$ is linear

Also higher order (of order 4)

Homogeneity Homogeneous as $a_0 = 0$.

Coefficients All constant at $a_x = 1 \; \forall x \in p$?

To solve, need to put into characteristic equation.

Use ansatz $X_n = A\lambda^n$

$$A\lambda^n = A\lambda^{n-1} + A\lambda^{n-2} + A\lambda^{n-4}$$

Divide through by $A\lambda^{n-4}$

$$A\lambda^n = A\lambda^{n-1} + A\lambda^{n-2} + A\lambda^{n-4} \text{ (repeated)}$$

$$\lambda^4 = \lambda^3 + \lambda^2 + 1$$ (recall $x^n \div x^m = x^n \times x^{-m} = x^{n-m}$)

Rearrange to equal 0

$$\lambda^4 - \lambda^3 - \lambda^2 - 1 = 0.$$

Solve (using calculator)

$\lambda = 1.755$ (4.s.f.)

-1

$0.1226 + 0.7449i$ (4.s.f.)

$0.1226 - 0.7449i$ (4.s.f.)

Stable iff ~~real part~~ ~~mod~~ modulus < 1.

Notes ~~a~~ scan has $(1.755)^t$ dominants.
Not quite sure what this means.

$X_n = nX_{n-1}$
Linearity: non-linear ×
Order: 1 ✓
Homogeneity: Homogeneous ✓
Coefficients: Not constant ✓

Apparently not scaled, as X_{n-1} doesn't get raised to any powers of n.

Can be solved Σ $X_n = n! X_0$.

COMP6216
[redacted]

~~Divide through by~~ (Whoops!)

$$X_n = \frac{X_{n-1}}{2} + 1$$

$$= X_{n-1} \times \tfrac{1}{2} + 1$$

Linearity: Linear
Order: 1
Homogeneity: Non-homogeneous
Coefficients: Constant

Homogeneous: $f(n) = 0 \; \forall n$. Or $a_0 = 0$.

<u>Solving</u>

Characteristic eqn not possible as non-homogeneous

[redacted] Missing temporal analysis.

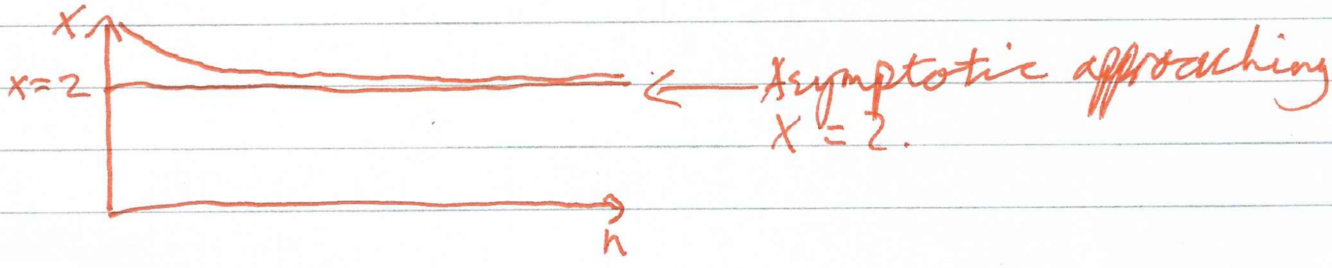
For $X_0 = 5$, $X_1 = 5/2 + 1 = 7/2 = 3.5$

$X_2 = 7/2 \times 1/2 + 1 = 11/4 = 2.75$

$X_3 = 11/4 \times 1/2 + 1 = 19/8 = 2.375$

$X \xrightarrow[\infty]{} 2$. Solutions mention $\lambda = 1/2$, which is the speed at which the solution converges.

Eg., $5 \rightarrow 3.5$ difference $3 \rightarrow 1.5$
$3.5 \rightarrow 2.75$ difference $1.5 \rightarrow 0.75$.



$\longrightarrow$ $X_n = 6X_{n-1} - 18X_{n-2}$

Order: 2 ✓

Linearity: Non Linear ✗

Homogenity: Homogeneous ✓

Coefficents: ~~Linear~~ Constant ✓

Linearity – strictly involves combination of X_{n-1} and a constant term.

Temporal Analysis –

As linear & homogeneous, it permits a characteristic equation. Ansatz:

$$X_n = A\lambda^n$$

$$A\lambda^n = 6A\lambda^{n-1} - 18A\lambda^{n-2} \quad (\div A\lambda^{n-2})$$

$$\lambda^2 = 6\lambda - 18 \quad \text{(rearrange to} = 0)$$

$$\lambda^2 - 6\lambda + 18 = 0$$

$$\lambda = 3 \pm 3i$$ ✓

Fluctuates as $|\lambda| > 1$ ✓ Update notes to clarify how to do this analysis.

Q3

$X_t = -2^{\cancel{t}} X_{t-1} + (-1)^t \quad X_0 = 1 \rightarrow -(2)^1(-1)^{1} \rightarrow -3$

$-3 \rightarrow -(2)^2 - 3 \rightarrow 12$

$x_0 = 0 \rightarrow -2^1(0) - 1 \rightarrow -1 \quad X_0 = -1$

$-1 \rightarrow 4(-1) + 1 \rightarrow$

Only order 2.

$X_t = -2^t X_{t-1} - 2^t X_{t-2} + (-1)^t$

$X_0 = 0 \quad X_1 = 0 \quad x_2 = +ve \quad x_3 = -ve$

$X_2 = -2^2(0) - 2^2(0) + (-1)^2 = 1$

$X_3 = -2^3(1) - 2^3(0) + (-1)^3 = -9$

$X_4 = -2^4(-9) - 2^4(1) + (-1)^4 = 123$

$144 \quad -216 \quad +1$

This works, but is messy. Importantly, it differs from the solution, as that does not address the 2nd order aspect of the question, nor handle conditions where X_0 is equal to zero. A simpler version would be

$X_t = -2X_{t-1} + 1.$

PTO

→ Exponential decline, positive & negative

2 complex conjugate λs such that $|\lambda| < 1$

$\lambda = \frac{1}{2} \pm \frac{1}{2} i$

$|\lambda| = \sqrt{2}/2 = \frac{1}{\sqrt{2}} < 1$

Into function

$(\lambda - \frac{1}{2}(1-i))(\lambda - \frac{1}{2}(1+i)) = 0$

$\lambda^2 - \frac{1}{2}\lambda(1-i) - \frac{1}{2}\lambda(1+i) + \frac{1}{4}(1-i)(1+i) = 0$

$\lambda^2 - \lambda(\frac{1}{2} \cancel{- \frac{1}{2}i} + \frac{1}{2} \cancel{+ \frac{1}{2}i}) + \frac{1}{4}(1-i)(1+i) = 0$

$\lambda^2 - \lambda + \frac{1}{4}(1 - i + i - i^2) = 0$

$+ \frac{1}{4}(1 - \sqrt{-1}^2)$

$+ \frac{1}{4}(1 - -1)$

$+ 2 = 0$

$X_n = -X_{n-1} + 2X_{n-2}$

→ Exp. increasing

$X_t = 2X_{t-2} + 1$

Q4

$X_n + X_{n-2} = 0 \quad \bar{c} \quad X_0 = 0 \quad X_1 = 1$

$X_n = -X_{n-2}$

$X_n = A\lambda^n$

$A\lambda^n = -A\lambda^{n-2} \quad (\div A\lambda^{n-2})$

$\lambda^2 = -1$

$\lambda = \pm i$

$X_t = c_1\lambda_1^t + c_2\lambda_2^t$

$0 = c_1\lambda_1^0 + c_2\lambda_2^0$

$= c_1 + c_2 \longrightarrow c_1 = -c_2$

$1 = c_1\lambda_1 + c_2\lambda_2$

$= c_1 i - c_2 i$ sub c_1 for $-c_2$

$1 = c_1 i + c_1 i$

$1 = 2c_1 i$

$c_1 = 1/2i$

$c_2 = -\frac{1}{2}i$

$\Rightarrow X_t = \frac{1}{2}i(i)^t - \frac{1}{2}i(-i)^t$

t is even $\Rightarrow 0 = X_{2k}$

odd $\Rightarrow$ sub in $t = 2k+1$

$X_{2k+1} = \frac{1}{2}i(i)^{2k+1} - \frac{1}{2}i(-i)^{2k+1}$

$= \frac{1}{2}i^2(i)^{2k} + \frac{1}{2}i^2(-i)^{2k}$

$= \frac{1}{2}(-1)(i)^{2k} + \frac{1}{2}(-1)(i)^{2k} \longrightarrow *$

$* = -\frac{1}{2}(i^2)^k - \frac{1}{2}(i^2)^k$

$= -(-1)^k$?

Solns a bit different.

Q5

$$X_n = \frac{X_{n-1}}{2} + 1$$

$$= \tfrac{1}{2} X_{n-1} + 1$$

TBC